TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 290, Number 2, August 1985

ON THE SPECTRA OF COMPACT NILMANIFOLDS

BY

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ABSTRACT. We show the equivalence of the Howe-Richardson multiplicity formula for compact nilmanifolds and the formula obtained by Corwin and Greenleaf using the Selberg trace formula.

Introduction. Let G be a connected simply connected nilpotent Lie group and suppose G contains a discrete cocompact subgroup Γ. Let $\rho = \operatorname{ind}_\Gamma^G(1)$. Then ρ is a direct sum of irreducible representations each occurring with finite multiplicity; we will write $\rho = \oplus m(\pi)\pi$. A basic problem in representation theory is to determine $m(\pi)$ and give a criterion for $m(\pi)$ not to be zero. Moore first studied this problem in **[M]** and later Howe **[H]** and Richardson **[R]** independently gave a closed formula for $m(\pi)$ that generalized the classical Frobenius reciprocity formula for finite groups. Using the Poisson summation and Selberg trace formulas, Corwin and Greenleaf **[C-G]** gave a formula for $m(\pi)$ that depended only on the coadjoint orbit in $\mathfrak{g}^*$ corresponding to π via Kirillov theory and the structure of Γ, but the connection between the two formulas was not clear. In §1 we consider the case when Γ is a lattice subgroup of G, i.e. $\log(\Gamma)$ is an additive subgroup of the Lie algebra of G, and show there is a simple relationship between the two. In §2 we show how Frobenius reciprocity can be used to reduce the general case to the lattice subgroup case.

Parts of this paper were in the author's doctoral dissertation and it is a pleasure to thank my advisor, Calvin Moore, for his support and guidance. I would also like to thank Larry Corwin for some useful conversation.

1. Let G be a connected simply connected nilpotent Lie group. We denote the Lie algebra of G by $\mathfrak{g}$ and the dual of g by g^*. Let exp: $\mathfrak{g} \to G$ be the exponential map and log: $G \to \mathfrak{g}$ its inverse. We let Ad be the adjoint action of G on $\mathfrak{g}$ and Ad^* the coadjoint on $\mathfrak{g}^*$. If π is an irreducible unitary representation we write $\mathcal{O}(\pi) \subseteq \mathfrak{g}^*$ for the coadjoint orbit associated to π via Kirillov theory. Let $\Gamma \subseteq G$ be a discrete cocompact subgroup of G. If Q denotes the rational numbers, then Γ determines a Q structure on $\mathfrak{g}$ by $\mathfrak{g}_Q = \operatorname{span}_Q\{\log(\Gamma)\}$. We say $g \in G$ is rational iff $g = \exp(X)$ with $X \in \mathfrak{g}_Q$ and let G_Q denote the set of rational points. Given a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ we say $\mathfrak{h}$ is rational if $\mathfrak{h} \cap \mathfrak{g}_Q$ contains a basis of $\mathfrak{h}$ over $\mathbf{R}$. This is equivalent to $\exp(\mathfrak{h}) = H$ having $H \cap \Gamma$ for a discrete cocompact subgroup. If $f \in \mathfrak{g}^*$, say, f is

Received by the editors May 30, 1984.
1980 *Mathematics Subject Classification*. Primary 22E25, Secondary 22E40, 22D30.

0002-9947/85 $1.00 + $.25 per page

rational if $f(\mathfrak{g}_Q) \subseteq Q$. For a very complete and detailed discussion of this see [**C-G**]. One should also note that if one views G as an affine algebraic group, then the existence of a cocompact Γ is equivalent to G being defined over Q and the notion of a rational point in the sense of algebraic groups is equivalent to the definition given above.

For the rest of this section we suppose that $\Lambda = \log(\Gamma)$ is an additive subgroup of $\mathfrak{g}$. Let $\Lambda^\perp = \{f \in \mathfrak{g}^* | f(\Lambda) \subseteq \mathbf{Z}\}$. Let π be an irreducible representation of G and $\mathcal{O} \subseteq \mathfrak{g}^*$ the coadjoint orbit corresponding to π. According to [**M**], we have $m(\pi) > 0$ if and only if $\mathcal{O} \cap \Lambda^\perp \neq \varnothing$, so we will suppose this intersection is nonempty. Since $\mathcal{O} \cap \Lambda^\perp$ is $\mathrm{Ad}^*(\Gamma)$ invariant we can write it as a union of $\mathrm{Ad}(\Gamma)$ orbits. To each such orbit $\Omega \subseteq \mathcal{O} \cap \Lambda^\perp$ one can associate a number $C(\Omega)$ as follows:

Let $f \in \Omega$ and $\mathfrak{g}(f) = \{X \in \mathfrak{g} | \mathrm{ad}^*(X)f = 0\}$. Then $\mathfrak{g}(f)$ is a rational subalgebra, so the $\mathbf{Z}$-rank of $\mathfrak{g}(f) \cap \Lambda$ is equal to $\dim(\mathfrak{g}(f))$. Choose a $\mathbf{Z}$-basis $X_1, \ldots, X_n$ of Λ (consequently it is an $\mathbf{R}$ basis for $\mathfrak{g}$) such that $X_1, \ldots, X_s$ span $\mathfrak{g}(f)$ over $\mathbf{R}$. Let A_f be the matrix with entries $f([X_i, X_j])$, $s < i, j \leqslant n$. Then $|\det(A_f)|$ is independent of the basis satisfying the above conditions and depends only on the Γ orbit of f in $\mathcal{O} \cap \Lambda^\perp$. Set $C(\Omega) = |\det(A_f)|^{-1/2}$. Then $C(\Omega)$ is a positive rational number. The multiplicity formula of Corwin and Greenleaf can be written as

$$m(\pi) = \sum_{\Omega \in \mathcal{O} \cap \Lambda^\perp / \mathrm{Ad}^*(\Gamma)} C(\Omega).$$

For details see [**C-G**].

We now describe the Howe-Richardson formula for $m(\pi)$. If $m(\pi) > 0$ there exists a rational element $f \in \mathcal{O}(\pi)$, fix such an element f. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a rational polarization for f and $H = \exp(\mathfrak{h})$ the connected subgroup of G with Lie algebra $\mathfrak{h}$. For $\mathfrak{h} \in H$ let $\chi_f(\mathfrak{h}) = \exp(2\pi i f(\log(\mathfrak{h})))$. Then χ_f is a character of H and π is equivalent to $\mathrm{ind}_H^G(\chi_f)$. Let $X = \{(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f) | g \in G\}$ and define an equivalence relation on X by $(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f) \sim (\mathrm{Ad}(g_1)\mathfrak{h}, \mathrm{Ad}^*(g_1)f)$ iff $\mathrm{Ad}(g)\mathfrak{h} = \mathrm{Ad}(g_1)\mathfrak{h}$ and $\mathrm{Ad}^*(g)f|_{\mathrm{Ad}(g)\mathfrak{h}} = \mathrm{Ad}^*(g_1)f|_{\mathrm{Ad}(g)\mathfrak{h}}$. Let $C(\mathfrak{h}, f)$ denote the set of equivalence classes. Define $L(\mathfrak{h}, f) \subseteq C(\mathfrak{h}, f)$ by $(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f) \in L(\mathfrak{h}, f)$ iff $\mathrm{Ad}(g)\mathfrak{h}$ is a rational subalgebra of $\mathfrak{g}$ and $\chi_{\mathrm{Ad}^*(g)f}|_{gHg^{-1} \cap \Gamma} \equiv 1$. There is a natural action of Γ on $L(\mathfrak{h}, f)$ by

$$\gamma \cdot \big(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f\big) = \big(\mathrm{Ad}(\gamma g)\mathfrak{h}, \mathrm{Ad}^*(\gamma g)f\big)$$

and the number of Γ orbits in $L(\mathfrak{h}, f)$ is the multiplicity $m(\pi)$. In what follows we will assume that $\mathfrak{h}$ and f have been chosen such that $\mathfrak{h}$ is rational and $\chi_f|_{H \cap \Gamma} \equiv 1$.

Suppose $\chi_{\mathrm{Ad}^*(g)f}|_{gHg^{-1} \cap \Gamma} \equiv 1$. Then $\mathrm{Ad}^*(g)f(\mathrm{Ad}(g)\mathfrak{h} \cap \Lambda) \subseteq \mathbf{Z}$: thus we can find $\phi \in (\mathrm{Ad}(g)\mathfrak{h})^\perp$ such that $\mathrm{Ad}^*(g)f + \phi \in \Lambda^\perp$. From $\mathrm{Ad}^*(gHg^{-1})(\mathrm{Ad}^*(g)f) = \mathrm{Ad}^*(g)f + (\mathrm{Ad}(g)\mathfrak{h})^\perp$ we have $\mathrm{Ad}^*(g)f + \phi = \mathrm{Ad}^*(y)(\mathrm{Ad}^*(g)f) = \mathrm{Ad}^*(yg)f$ for some $y \in gHg^{-1}$. Since $(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f)$ is equivalent to $(\mathrm{Ad}(yg)\mathfrak{h}, \mathrm{Ad}^*(yg)f)$ we have that every equivalence class in $L(\mathfrak{h}, f)$ has a representative $(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f)$ such that $\mathrm{Ad}^*(g)f \in \Lambda^\perp$. We can now define a surjective map α: $\mathcal{O} \cap \Lambda^\perp \to L(\mathfrak{h}, f)$. If $\phi \in \Lambda^\perp$, then both ϕ and f are rational points in $\mathcal{O}$; it follows [**B** or **M**] that there exists $g \in G_Q$, the rational points of G, such that $\mathrm{Ad}^*(g)f = \phi$. Define $\alpha(\phi) = \alpha(\mathrm{Ad}^*(g)f) = (\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f)$. Since g is rational, $\mathrm{Ad}(g)\mathfrak{h}$ is a rational subalgebra of $\mathfrak{g}$ and $\phi \in \Lambda^\perp$ automatically says $\chi_{\mathrm{Ad}^*(g)f}|_{gHg^{-1}\cap\Gamma} \equiv 1$. Let $G(f) = \{g \in G | \mathrm{Ad}^*(g)f = f\}$; since $G(f) \subseteq H$ for any polarization H we see that α is well defined. Since α is a Γ equivariant map we can write the Corwin-Greenleaf formula as follows:

$$m(\pi) = \sum_{\omega \in L(\mathfrak{h}, f)/\mathrm{Ad}^*(\Gamma)} \sum_{\Omega \in \alpha^{-1}(\omega)} C(\Omega).$$

The equivalence of the Howe-Richardson formula and the Corwin-Greenleaf formula for Γ a lattice subgroup follows from above once we show

THEOREM 1. *With notation as above* $\sum_{\Omega \in \alpha^{-1}(\omega)} C(\Omega) = 1$.

Before we begin with the proof of Theorem 1 we need the following lemmas from [**H** and **C-G**].

LEMMA 1. *Let $\Gamma \subseteq G$ be a discrete cocompact subgroup of G, suppose Z = center of $\mathfrak{g}$ is one-dimensional and let $z \in \log(\Gamma) \cap Z$ be a generator. Then there exists $y \in \mathfrak{g}$ such that if W = span of y and z over $\mathbf{R}$, then y and z generate $\log(\Gamma) \cap W$. Let $\mathfrak{g}_1$ = the centralizer of y in $\mathfrak{g}$. Then $\mathfrak{g}_1$ is rational and of codimension 1 in $\mathfrak{g}$. $\mathfrak{g}_1$ is the Kirillov codimension 1 subalgebra* [**K**]. *There is $x \in \log(\Gamma)$ such that if $\Gamma_1 = \Gamma \cap \exp(\mathfrak{g}_1)$, then Γ_1 and $\exp(x)$ generate Γ. If L = span over $\mathbf{R}$ of x, y, z, then L is a three-dimensional Heisenberg algebra and $\exp(x)$, $\exp(y)$, $\exp(z)$ generate $\exp(L) \cap \Gamma$. Finally, there exists $a \in \mathbf{Z}$, $a \neq 0$, so that $[x, y] = az$ and a is independent of the choice of x satisfying the above conditions.*

DEFINITION 1. Let Γ be a discrete torsion-free nilpotent group. A weak Malcev basis for Γ is a set $\{d_1, \dots, d_p\} \subseteq \Gamma$ such that:

(i) For any $d \in \Gamma$ there is a decomposition $d \equiv d_1^{n_1} \cdots d_p^{n_p}$, where $n_i \in \mathbf{Z}$.

(ii) The set $\Gamma_i = d_1^{\mathbf{Z}} \cdots d_i^{\mathbf{Z}}$ is a subgroup with Γ_{i-1} normal in Γ_i for $i = 1, 2, \dots, p$.

(iii) $\Gamma_i/\Gamma_{i-1} \approx \mathbf{Z}$ for $i = 2, \dots, p$.

DEFINITION 2. Let G be a simply connected nilpotent Lie group. A weak Malcev basis for G is a set $\{X_1, \dots, X_p\} \subseteq \mathfrak{g}$ such that:

(i) For $X \in G\ \exists t_i \in \mathbf{R}$, $i - 1, \dots, p$, such that $X = \gamma_1(t_1) \cdots \gamma_p(t_p)$, where $\gamma_i(t) = \exp(tX_i)$.

(ii) The set $G_i = \gamma_1(\mathbf{R}) \cdots \gamma_i(\mathbf{R})$ is a closed subgroup of G with G_{i-1} normal in G_i for each i.

(iii) $G_i/G_{i-1} \approx \mathbf{R}$.

Weak Malcev basis is an adaptation of Malcev's coordinates of the 2nd kind as necessitated by inducing from nonnormal subgroups in the Kirillov model [**Ma**].

If $\Gamma \subseteq G$ is a discrete cocompact subgroup of G we say a weak Malcev basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ is subordinate to Γ if $\{\exp(X_1), \dots, \exp(X_n)\}$ is a weak Malcev basis for Γ.

LEMMA 2. *Let* Γ *be a discrete cocompact subgroup of* G; *if* $\{d_1,\dots,d_n\}$ *is a weak Malcev basis for* Γ, *then* $\{X_i = \log(d_i) | i = 1,\dots,n\}$ *is a weak Malcev basis of* G *subordinate to* Γ.

LEMMA 3. *Let* G *and* Γ *be as above*, $M \subseteq G$ *a closed connected subgroup of* G *such that* $M/M \cap \Gamma$ *is compact. If* $\{X_1,\dots,X_s\}$ *is a weak Malcev basis of* M *subordinate to* $M \cap \Gamma$, *then it can be extended to a weak Malcev basis of* G *subordinate to* Γ_0.

LEMMA 4. *If* Γ *is a lattice subgroup of* G *and* $\{X_1,\dots,X_n\}$ *is a weak Malcev basis of* G *subordinate to* Γ, *then* $\{X_1,\dots,X_n\}$ *forms a* $\mathbf{Z}$ *basis of* $A = \log(\Gamma)$.

PROOF OF THEOREM 1. We proceed by induction on $\dim(G)$. If $\dim(G) = 1$ the statement is trivial and therefore suppose $\dim(G) > 1$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$; if $\dim(\ker(f) \cap \mathfrak{z}) > 1$ we can divide out the corresponding central subgroup and proceed to a lower dimensional case. Consequently we can assume $\dim(\mathfrak{z}) = 1$, and if x, y, z and $\mathfrak{g}_1$ are as in Lemma 1, then $f(z) = \lambda \neq 0$. Note that for $f_1, f_2 \in \mathcal{O}(\pi)$, $f_1(z) = \mathrm{Ad}^*(g)f_2(z) = f_2(z) = \lambda$.

We can assume $\omega = \Gamma \cdot (\mathfrak{h}, f)$ simply by relabeling. To find Γ orbits in $\mathcal{O} \cap \Lambda^\perp$ such that $\alpha(\Omega) = \omega$ we proceed as follows: The point $(H, f) \in \omega$ has $\Gamma \cap H$ for its stability group; consequently $\Gamma \cap H$ preserves the fiber $\alpha^{-1}((H, f)) = (f + \mathfrak{h}^\perp) \cap \Lambda^\perp$. A set of representatives for Γ orbits in $\mathcal{O} \cap \Lambda^\perp$ that map into ω via α is given by a set of representatives of $\Gamma \cap H$ orbits in $(f + \mathfrak{h}^\perp) \cap \Lambda^\perp$.

As usual, there are two cases we must consider.

Case I. Suppose $\mathfrak{h} \subseteq \mathfrak{g}_1$. Let $\bar{f}$ be the restriction of f to $\mathfrak{g}_1$. Then $\mathfrak{h}$ is a rational polarization for $\bar{f}$ and $\bar{\pi} = \mathrm{ind}_H^{G_1}(\chi_{\bar{f}})$ occurs in $L^2(G_1/\Gamma_1)$ by the criterion in [**C-G** or **M**], i.e., $\bar{f} \in \Lambda_1^\perp \subseteq \mathfrak{g}_1^*$. Let $\mathfrak{h}^{\perp 1} = \{\phi \in \mathfrak{g}_1^* | \phi(x) = 0 \ \forall X \in \mathfrak{h}\}$ and let $r\colon \mathfrak{g}^* \to \mathfrak{g}_1^*$ be the restriction map—so r is Ad^* equivariant. If $L = (f + \mathfrak{h}^\perp) \cap \Lambda^\perp$ and $L_1 = (\bar{f} + h^{\perp 1}) \cap \Lambda_1^\perp$, then $r|_L\colon L \to L_1$ is a $\Gamma \cap H = \Gamma_1 \cap H$ equivariant surjective map. If $S_1 \subseteq L_1$ is a set of $\Gamma \cap H$ orbit representatives, then $(r|_L)^{-1}(S_1)$ contains a set of $\Gamma \cap H$ orbit representatives in L, say S. Let $\bar{\phi} \in S_1$, $\phi \in (r|_L)^{-1}(\bar{\phi})$, $G_1(\bar{\phi})$ = stability group of $\bar{\phi}$ in G, and $G(\phi)$ be the stability group of ϕ in G. Then $G_1(\bar{\phi}) = G(\phi) \cdot \{\exp(tY) | t \in \mathbf{R}\}$. Consequently a $\Gamma \cap G_1(\bar{\phi})$ orbit in $(r|_L)^{-1}(\bar{\phi})$ is the same as a $\Gamma \cap \{\exp(tY) | t \in \mathbf{R}\} = \{\exp(nY) | n \in \mathbf{Z}\}$ orbit in $(r|_L)^{-1}(\phi)$. An element $\phi \in (r|_L)^{-1}(\bar{\phi})$ is determined by its value on $x \in \mathfrak{g}$. Suppose $\phi(x) = b$. Then $\mathrm{Ad}^*(\exp(nY))(\phi)(x) = b + na\lambda$ and we see there are $|a\lambda| \{\exp(nY) | n \in \mathbf{Z}\}$ orbits in $(r|_L)^{-1}(\bar{\phi})$.

Let $\phi \in (r|_L)^{-1}(\bar{\phi})$ for some $\bar{\phi} \in S_1$. We want to compute $C(\Gamma \cdot \phi)$. To do this we need a basis $X_1,\dots,X_n$ of $\mathfrak{g}$ such that the $\mathbf{Z}$ span is Λ and $X_1,\dots,X_s$ span $\mathfrak{g}(\phi)$ over $\mathbf{R}$. Using Lemmas 3 and 4 we can find a basis $X_1 = z$, $X_2 = y,\dots,X_{n-1}$ of $\mathfrak{g}_1$ such that $X_1,\dots,X_{n-1}$ span Λ_1 over $\mathbf{Z}$, $X_1, X_2,\dots,X_s$ span $\mathfrak{g}_1(\bar{\phi})$, and $X_1, X_3, X_4,\dots,X_s$ span $\mathfrak{g}(\phi) \subseteq \mathfrak{g}_1(\bar{\phi})$. We have $X_1, X_3,\dots,X_{n-1}$ y, x span Λ over $\mathbf{Z}$ and $X_1, X_3,\dots,X_s$ span $\mathfrak{g}(\phi)$. Using this basis we can compute $C(\Gamma \circ \phi) = C(\phi)$. Recall that $C(\phi) = |\det(A(\phi))|^{1/2}$, where $A(\phi) = (\phi([x_i, x_j]))$, $s \leqslant i, j \leqslant n$. Since $[Y, \mathfrak{g}_1] = 0$, if we expand $A(\phi)$ on that row and column, using $\phi([x, y]) = \lambda a$, we get $\det(A(\phi)) = |\lambda a|^2 \det(A(\bar{\phi}))$. Consequently, we have $C(\phi) = |\lambda a|^{-1} \cdot C(\bar{\phi})$ for all $\bar{\phi} \in S_1$.

By our induction hypothesis $\Sigma_{\bar{\phi}\in S_1} C(\bar{\phi}) = 1$, so we get

$$\sum_{\phi\in S} C(\phi) = \sum_{\bar{\phi}\in S_1}\left(\sum_{\phi\in (r|_L)^{-1}(\bar{\phi})\cap S} C(\phi)\right)$$

$$= \sum_{\bar{\phi}\in S_1} C(\bar{\phi})\left(\sum_{\phi\in (r|_L)^{-1}(\bar{\phi})\cap S} |\lambda a|^{-1}\right) = \sum_{\bar{\phi}\in S_1} C(\bar{\phi}) = 1.$$

Thus Case I is verified.

Case II. Now suppose $\mathfrak{h} \not\subseteq \mathfrak{g}_1$ and set $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_1$, $\bar{\mathfrak{h}} = \operatorname{span}(\mathfrak{h}_0, Y)$. Then $\bar{\mathfrak{h}} \subseteq \mathfrak{g}_1$ and is a polarization for f. As before, let $r\colon g^* \to \mathfrak{g}_1^{*\prime}$ be the restriction map. Then $r|_{f+\mathfrak{h}^\perp}\colon f + \mathfrak{h}^\perp \to \mathfrak{g}_1^*$ is injective and $r(f + \mathfrak{h}^\perp) = \bigcup_{s\in\mathbf{R}}(f + sy^* + \bar{\mathfrak{h}}^{\perp 1})$. Consequently we have $r((f + \mathfrak{h}^\perp) \cap \Lambda^\perp) = \bigcup_{n\in\mathbf{Z}}(f + ny^* + \mathfrak{h}^{\perp 1}) \cap \Lambda_1^\perp$. A set S of $\Gamma \cap H$ orbit representatives in $(f + \mathfrak{h}^\perp) \cap \Lambda^\perp$ can be written as $\bigcup_{0\leqslant b<|\lambda a|} S_b$, where S_b is a set of $\Gamma \cap \bar{H}$ orbit representatives in $(f + by^* + \bar{\mathfrak{h}}^{\perp 1}) \cap \Lambda_1^\perp$ and $b \in \mathbf{Z}$. As before, if $\phi \in r^{-1}(S_b) \cap \Lambda^\perp$, then $C(\phi) = |\lambda a|^{-1} C(\bar{\phi})$, where $\bar{\phi} = r(\phi)$. Thus we have

$$\sum_{\phi\in S} C(\phi) = \sum_{0\leqslant b<|\lambda a|}\left(\sum_{\phi\in r^{-1}(S_b)\cap S} C(\phi)\right)$$

$$= \sum_{0\leqslant b<|\lambda a|} |\lambda a|^{-1}\left(\sum_{\bar{\phi}\in S_b} C(\bar{\phi})\right) = \sum_{0\leqslant b<|\lambda c|} |\lambda a|^{-1} = 1. \qquad \text{Q.E.D.}$$

COROLLARY 1 (MOORE [M]). $m(\pi) \leqslant \#\{\mathcal{O} \cap \Lambda^\perp/\mathrm{Ad}^*(\Gamma)\}$.

PROOF. By the above proof, $C(\phi)^{-1} = |\lambda a| C(\bar{\phi})$; thus we can reason by induction to conclude that $C(\phi)^{-1}$ is an integer. Q.E.D.

In [**R**] Richardson constructed a polarization for f such that in the inductive reasoning one never has Case II occurring. We note that if $\mathfrak{h}$ is a Richardson polarization, then the above proof shows that the $C(\Omega)$s are the same for every Ω such that $\alpha(\Omega) = \omega$. Since $\Sigma_{\alpha(\Omega)=\omega} C(\Omega) = 1$, $C(\Omega)$ equals the number of $\Gamma \cap H$ orbits in $f + \mathfrak{h}^\perp \cap \Lambda^\perp$. This observation was pointed out to me by Larry Corwin.

2. If π is a unitary representation of G on the Hilbert space $H(\pi)$, we let $H^\infty(\pi) = \{u \in H(\pi) | q \to \pi(q)u \text{ is a } C^\infty\text{-mapping}\}$. There is a representation of $\mathfrak{g}$, the Lie algebra of G, on $H^\infty(\pi)$. If $X \in \mathfrak{g}$, $u \in H^\infty(\pi)$ define

$$\pi(X)u = \frac{d}{dt}\left(\pi(\exp(tX)u)\right)\Big|_{t=0}.$$

Then $X \to \pi(X)$ is a Lie algebra representation of $\mathfrak{g}$, so it extends to a representation of $\mathfrak{U}(\mathfrak{g})$. Given $a \in \mathfrak{U}(\mathfrak{g})$ define a seminorm on $H^\infty(\pi)$ by $\rho_a(u) = \|\pi(a)u\|$. Then $H^\infty(\pi)$ has the structure of a Fréchet space with respect to these seminorms and we let $H^{-\infty}(\pi)$ be the topological dual of $H^\infty(\pi)$. For details see [**P-1**]. We write π^∞ for the restriction of π to $H^\infty(\pi)$ and $\pi^{-\infty}$ for the dual representation of G on $H^{-\infty}(\pi)$. The following is in [**P-1**].

THEOREM 2.1 (PENNEY). *Let G be a Lie group, Γ a discrete cocompact subgroup and π an irreducible representation of G. Then*

$$\mathrm{Hom}_G\big(\pi, \mathrm{ind}_\Gamma^G(1)\big) \simeq \mathrm{Hom}_\Gamma(1, \pi^{-\infty}).$$

If we set $\Gamma(\pi) = \{D \in H^{-\infty}(\pi) | \pi^{-\infty}(\lambda)D = D\ \forall\lambda \in \Gamma\}$, then the above theorem says $\dim(\mathrm{Hom}_G(\pi, \mathrm{ind}_\Gamma^G(1))) = \dim(\Gamma(\pi))$.

When G is a connected simply connected nilpotent Lie group, the lift maps of Richardson can be viewed as providing a basis of $\Gamma(\pi)$ **[R]**. If $\pi = \mathrm{ind}_H^G(\chi_f)$, then $H^\infty(\pi)$ corresponds to all Schwartz functions on G/H **[K]** and for each Γ orbit in $L(\mathfrak{h}, f)$ one can construct an element of $\Gamma(\pi)$. If $(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f)$ is a point of $L(\mathfrak{h}, f)$, then we can construct $D_g \in H^{-\infty}(\pi)$ as follows: For $\phi \in H^\infty(\pi)$ let $D_g(\phi) = \sum_{\gamma \in \Gamma/\Gamma \cap gHg^{-1}} \phi(\gamma g)$. The D_g's are linearly independent for g's in different $\Gamma : H$ double cosets and they span $\Gamma(\pi)$ **[P-2, F]**.

Now suppose $\Gamma_0 \subseteq \Gamma$ is a normal subgroup of finite index and we know the truth of the Howe-Richardson formula for Γ_0.

Let $L_0(\mathfrak{h}, f)$ be defined using Γ_0 and $L(\mathfrak{h}, f)$ be defined using Γ. Of course $L(\mathfrak{h}, f) \subseteq L_0(\mathfrak{h}, f)$, so we will suppose $L_0(\mathfrak{h}, f)$ is not empty. Let $D_g \in \Gamma_0(\pi) \subseteq H^{-\infty}(\pi)$. For $\gamma \in \Gamma$, $\phi \in H^\infty(\pi)$ we have

$$\big(\pi^{-\infty}(\gamma)D_g\big)(\phi) = \sum_{\delta \in \Gamma_0/\Gamma_0 \cap gHg^{-1}} \phi(\gamma\delta g) = \sum_{\delta \in \Gamma_0/\Gamma_0 \cap \gamma gHg^{-1}\gamma^{-1}} \phi(\delta\gamma g).$$

Thus $\pi^{-\infty}(\gamma)$ stabilizes $\mathbf{C} \cdot D_g$ iff $\gamma \in \Gamma_0 \cdot (\Gamma \cap gHg^{-1})$, then

$$\pi^{-\infty}(\gamma)D_g = \chi_f(g^{-1}\gamma^{-1}g)D_g = \overline{\chi}_{g\cdot f}(\gamma) \cdot D_g$$

(where $\overline{\chi}_{g\cdot f}$ extends to a character of $\Gamma_0 \cdot (\Gamma \cap gHg^{-1})$ by being trivial on Γ_0). If we set $W_g = \mathrm{span}\{\pi^{-\infty}(\gamma)D_g | \gamma \in \Gamma\}$, then we see that $\pi^{-\infty}|_\Gamma$ acting on W_g is exactly $\mathrm{ind}_{\Gamma_0\cdot(\Gamma \cap gHg^{-1})}^\Gamma(\overline{\chi}_{q\cdot f})$. Let S be a set of representatives for Γ orbits in $L_0(\mathfrak{h}, f)$, so given $g \in S$ we get $(\mathrm{Ad}(g)\mathfrak{h}, \mathrm{Ad}^*(g)f)$ or equivalently a $D_g \in H^{-\infty}(\pi)$. From above we see that the representation $\pi^{-\infty}|_\Gamma$ acting on $\Gamma_0(\pi)$ is

$$\bigoplus_{g \in S} \mathrm{ind}_{\Gamma_0\cdot(\Gamma \cap gHg^{-1})}^\Gamma\big(\overline{\chi}_{g\cdot f}\big).$$

Since $\Gamma(\pi) \subseteq \Gamma_0(\pi)$, we have that

$$nn\mathrm{Hom}_\Gamma(1, \pi^{-\infty}) \approx \bigoplus_{g \in S} \mathrm{Hom}_\Gamma\Big(1, \mathrm{ind}_{\Gamma_0\cdot(\Gamma \cap gHg^{-1})}^\Gamma\big(\overline{\chi}_{q\cdot f}\big)\Big)$$
$$\approx \bigoplus_{g \in S} \mathrm{Hom}_{\Gamma_0\cdot(\Gamma \cap gHg^{-1})}\big(1, \overline{\chi}_{g\cdot f}\big).$$

Thus

$$\dim\Big(\mathrm{Hom}_\Gamma\Big(1, \mathrm{ind}_{\Gamma_0\cdot(\Gamma \cap gHg^{-1})}^\Gamma\big(\overline{\chi}_{g\cdot f}\big)\Big)\Big) = \begin{cases} 1 & \text{if } \chi_{g\cdot f}|_{gHg^{-1}\cap\Gamma} \equiv 1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally those $g \in S$ such that $\chi_{g\cdot f}|_{gHg^{-1}\cap\Gamma} \equiv 1$ are parametrized by $L(\mathfrak{h}, f)$. Thus we have

PROPOSITION. *If $\Gamma_0 \subseteq \Gamma$ is a normal subgroup and the Howe-Richardson formula for Γ_0 is known, then the Howe-Richardson formula for Γ is true.*

REMARK. It is shown in [**M** or **C-G**] that for a given Γ it is always possible to find a normal Γ_0 such that Γ_0 is a lattice subgroup.

REFERENCES

[**B**] A. Borel, *Linear algebraic groups*, Benjamin, New York, 1969.

[**C-G**] L. Corwin and F. Greenleaf, *Character formulas and spectra of compact nilmanifolds*, J. Funct. Anal. **21** (1976), 123–154.

[**F**] J. Fox, *Frobenius reciprocity and extensions of nilpotent Lie groups* (preprint 1984).

[**H**] R. Howe, *On Frobenius reciprocity for unipotent algebraic groups over Q*, Amer. J. Math. **93** (1971), 163–172.

[**K**] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys **17** (1962), 53–104.

[**Ma**] A. Malcev, *On a class of homogeneous spaces*, Amer. Math. Soc. Transl., ser. 1, no. 39 (1951), 33 pp.

[**M**] C. C. Moore, *Decomposition of unitary representations defined by discrete subgroups of nilpotent groups*, Ann. of Math. (2) **82** (1965), 146–182.

[**P-1**] R. Penney, *Abstract Plancherel theorems and a Frobenius reciprocity theorem*, J. Funct. Anal. **18** (1975), 177–190.

[**P-2**] R. Penney, F. Greenleaf and L. Corwin, *A general character formula for irreducible projections on L^2 of a nilmanifold*, Math. Ann. **225** (1977), 21–32.

[**R**] L. Richardson, *Decomposition of the L^2 space of a general compact nilmanifold*, Amer. J. Math. **93** (1971), 173–190.

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